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Periodic Locally Soluble Groups with the Minimal Condition on Centralizers

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INTRODUCTION

In a recent paper [1] the first author studied locally finite groups satisfying the minimal condition on centralizers of subsets. One of the main results was that, for every prime p, the Sylow p-subgroups of such a group are conjugate. In the present paper it is shown that if G is a periodic soluble group satisfying the minimal condition on centralizers, then, for every set π of primes, the Sylow π-subgroups (by which we simply mean maximal π-subgroups) of G are conjugate. It follows that G belongs to the class $\mathfrak{U}$ introduced in [4], and hence the structure of G is constrained by the results of [5] and [8]. In this vein we shall show that G is nilpotent-by-Abelian-by-finite. Our other main result is that a periodic locally soluble group satisfying the minimal condition on centralizers is soluble. These results are generalizations of known results for linear groups (see [13]), and, more generally, for CZ-groups (see [2, 3, 9, 10]). However we conclude with an example to show the limitations of the comparison with linear groups.

1. SYLOW SUBGROUPS

Throughout this paper π will denote a set of primes and π' the complementary set. Further notation and terminology is as in [1]. In particular, $\mathfrak{M}_c$ denotes the class of all groups satisfying the minimal condition on centralizers. We extend

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this notation by writing $\mathfrak{M}_c^\pi$ for the class of all groups G such that for every π'-subgroup H and every π-subgroup K the minimal condition is satisfied for centralizers in H of subgroups of K. Clearly, if G belongs to $\mathfrak{M}_c^\pi$ then so does every subgroup of G.

LEMMA 1.1. *The following conditions on a group G are equivalent.*

(i) $G \in \mathfrak{M}_c^\pi$.

(ii) *For every π'-subgroup H of G and every π-subgroup K of G there exists a finite subset K_0 of K such that $C_H(K) = C_H(K_0)$.*

(iii) *For every π'-subgroup H of G and every π-subgroup K of G the maximal condition is satisfied for centralizers in K of subgroups of H.*

Proof. It is easy to prove that (i) and (ii) are equivalent. Let H be a π'-subgroup of G and K a π-subgroup of G. Then, for every subset S of H and every subset T of K,

$$C_K(S) = C_K(C_H(C_K(S))), \qquad C_H(T) = C_H(C_K(C_H(T))).$$

This yields that (i) and (iii) are equivalent.

The next result is essentially a corollary of the results of [5, 6]. A slightly weaker version of it has already been noted in [10]. It is an extension of the Schur–Zassenhaus theorem for finite groups.

THEOREM 1.2. *Let G be a locally finite $\mathfrak{M}_c^\pi$-group with a normal π'-subgroup N such that G/N is a π-group. Then N has complements in G. The complements are all conjugate and are precisely the Sylow π-subgroups of G.*

Proof. It is enough to prove that G splits over N and that the Sylow π-subgroups of G are conjugate. Thus, by Lemmas 2.3 and 2.1 of [6], it is enough to prove that, for every countable subgroup G_0 of G, the Sylow π-subgroups of G_0 are conjugate. By Lemma 2.1 of [5] and the Schur–Zassenhaus theorem for finite groups (the Feit–Thompson theorem being needed for the conjugacy of complements in general), G_0 splits over $G_0 \cap N$. The result follows by Lemma 4.3 of [5].

LEMMA 1.3. *Let G be a locally finite $\mathfrak{M}_c^\pi$-group and let N be a normal π'-subgroup of G. Let A/N be a π'-subgroup of G/N and B_1 a π-subgroup of G such that $[A/N, B_1N/N] = 1$. Then $A = C_A(B_1)N$.*

Proof. B_1N is a normal subgroup of B_1A. By Theorem 1.2 the Sylow π-subgroups of B_1N are all conjugate in B_1N to B_1. Thus, by the Frattini argument,

$$B_1A = N_{B_1A}(B_1)\, B_1N = N_{B_1A}(B_1)N.$$

Thus $A = N_A(B_1)N$. But $A \cap B_1 = 1$ and A is normalized by B_1. Thus $N_A(B_1) = C_A(B_1)$, giving $A = C_A(B_1)N$, as required.

A group G is called *π-separable* if there is a finite series, called a *π-series*,

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

of subgroups of G such that each G_i/G_{i-1} is either a π-group or a π'-group. If N is π-separable and N is a normal subgroup of a group G then there is a π-series of N consisting of normal subgroups of G.

LEMMA 1.4. *Let G be a locally finite group belonging to $\mathfrak{M}_c{}^{\pi} \cap \mathfrak{M}_c^{\pi'}$ and let N be a π-separable normal subgroup of G.*

(i) $G/N \in \mathfrak{M}_c{}^{\pi} \cap \mathfrak{M}_c^{\pi'}$.

(ii) *Suppose that A/N is a π'-subgroup of G/N and B/N a π-subgroup of G/N such that $[A/N, B/N] = 1$. Then there exists a π'-subgroup A_1 of G and a π-subgroup B_1 of G such that $A_1N = A$, $B_1N = B$, and $[A_1, B_1] = 1$.*

Proof. (i) We may assume that N is a π'-group. Let H/N be a π'-subgroup of G/N and K/N a π-subgroup of G/N. By Theorem 1.2 there exists a complement K_1 for N in K. Let A/N be any subgroup of H/N and let $B/N = C_{K/N}(A/N)$. Thus $B = B_1N$, where $B_1 = B \cap K_1$. Let $A_1 = C_A(B_1)$. Then, clearly, $B_1 \leqslant C_{K_1}(A_1)$. But, by Lemma 1.3, $A = A_1N$. Thus

$$C_{K_1}(A_1)N/N \leqslant C_{K/N}(A/N) = B/N = B_1N/N.$$

Hence $C_{K_1}(A_1) \leqslant B_1$ and so $B_1 = C_{K_1}(A_1)$. It follows that the map sending B/N to B_1 is a one-one order-preserving map from the set of centralizers in K/N of subgroups of H/N into the set of centralizers in K_1 of subgroups of H. By hypothesis, and Lemma 1.1, the second set satisfies the maximal and minimal conditions. Hence so does the first; and the result follows.

(ii) By (i), we may assume that N is a π'-group. By Theorem 1.2, there exists a π-subgroup B_1 of G such that $B_1N = B$. By Lemma 1.3 it suffices to take $A_1 = C_A(B_1)$.

LEMMA 1.5. *Let G be a π-separable locally finite group belonging to $\mathfrak{M}_c{}^{\pi} \cap \mathfrak{M}_c^{\pi'}$. Then every factor group of every subgroup of G belongs to $\mathfrak{M}_c{}^{\pi} \cap \mathfrak{M}_c^{\pi'}$. The Sylow π-subgroups of G are conjugate.*

Proof. It is trivial that the hypotheses are inherited by subgroups of G and Lemma 1.4 shows that the hypotheses are inherited by factor groups of subgroups. The fact that the Sylow π-subgroups of G are conjugate now follows from Lemma 2.1 of [6] and Lemma 5.1 of [5]. Alternatively this can be established by Theorem 1.2 and induction on the length of a π-series of G.

THEOREM 1.6. *Let G be a periodic soluble $\mathfrak{M}_c$-group. Then, for every set π of primes, the Sylow π-subgroups of G are conjugate.*

Proof. For all π, G satisfies the hypotheses of Lemma 1.5.

Let $\mathfrak{U}$ be the class of groups defined in [4]. By results from [6] (see Theorem E and neighboring remarks), $\mathfrak{U}$ may be described as the class of all locally finite groups G such that, for every set π of primes, the Sylow π-subgroups of every subgroup of G are conjugate. By Theorem 1.6, $\mathfrak{U}$ contains every periodic soluble $\mathfrak{M}_c$-group. Indeed, by Lemma 1.5, $\mathfrak{U}$ contains every locally finite group which is π-separable and belongs to $\mathfrak{M}_c^{\pi}$ for all π.

2. LOCALLY SOLUBLE GROUPS

In this section we shall describe the structure of periodic soluble $\mathfrak{M}_c$-groups and show that periodic locally soluble $\mathfrak{M}_c$-groups are soluble.

THEOREM 2.1. *Let G be a periodic soluble $\mathfrak{M}_c$-group. Then G has a nilpotent normal subgroup N such that G/N is Abelian-by-finite of finite (Prüfer) rank.*

Proof. By Theorem 1.6, $G \in \mathfrak{U}$. Let $\rho(G)$ be the Hirsch–Plotkin radical of G. Then, by Theorem C of [8], and in the terminology of [8], $G/\rho(G)$ has a subgroup of finite index which is a subdirect product of finitely many "pinched" groups. The pinched groups arising belong to $\mathfrak{M}_c^{\pi}$ for all π by Lemma 1.5.

Let P be a pinched group which belongs to $\mathfrak{M}_c^{\pi}$ for all π. By Lemma 6.5 of [8], P is a semidirect product $P = AB$ ($B \lhd P$, $A \cap B = 1$) of locally cyclic subgroups A and B such that, for some set π of primes, A is a π'-group and B is a π-group. For each prime p let B_p be the Sylow p-subgroup of the Abelian group B. Since $P \in \mathfrak{M}_c^{\pi}$ there exists a finite subset $\{p_1, ..., p_k\}$ of π such that $C_A(B) = C_A(B_{p_1} \times \cdots \times B_{p_k})$. But $A/C_A(B_{p_1} \times \cdots \times B_{p_k})$ is finite (by Theorem 1. F.3 of [11], for example). It follows that P is Abelian-by-finite of finite rank.

Therefore, $G/\rho(G)$ is Abelian-by-finite of finite rank. Let $H/\rho(G)$ be an Abelian subgroup of finite index in $G/\rho(G)$. By Theorem A of [1], $\rho(G)$ is nilpotent-by-finite. Thus $\rho(G)$ has a nilpotent subgroup N of finite index which is normal in G. It follows that G/N has finite rank. Let $K = H/N$. Then K' is finite. Thus $C_K(K')$ has finite index in K and is nilpotent of class at most 2. For each prime p let S_p be the Sylow p-subgroup of $C_K(K')$. Since $C_K(K')$ has finite rank, each S_p is Abelian-by-finite (by Lemma 1.E.7 of [11], for example). But S_p is Abelian for all p not dividing the order of K'. It follows that K is Abelian-by-finite. Hence G/N is Abelian-by-finite and the proof is complete.

THEOREM 2.2. *Let G be a periodic locally soluble $\mathfrak{M}_c$-group. Then G is soluble.*

Proof. Since $G \in \mathfrak{M}_c$ we may assume that every centralizer in G which is a proper subgroup of G is soluble. We shall first prove that G is π-separable for all π.

Let $\rho(G)$ be the Hirsch–Plotkin radical of G. If $O_{p,p'}(G) = G$ for all odd primes p then $G/\rho(G)$ is a 2-group and it follows that G is π-separable for all π. Thus we may assume that there is an odd prime p such that $O_{p,p'}(G) < G$. Let $N = O_p(G)$ and $K = O_{p,p'}(G)$. Since G/N is not a p'-group it has a subgroup A/N of order p. Let $B/N = C_{K/N}(A/N)$. Then by Lemma 1.4 there is a p-subgroup A_1 of G and a p'-subgroup B_1 of G such that $A_1N = A$, $B_1N = B$, and $[A_1, B_1] = 1$. Since $A_1 \nleq N$, $C_G(A_1) < G$. Thus $C_G(A_1)$ is soluble. Thus B_1 is soluble. Therefore B/N is soluble.

Let d be the derived length of B/N. Then, for every finite A/N-invariant subgroup F/N of K/N, $C_{F/N}(A/N)$ has derived length at most d. By the corollary of [12], it follows that F/N has Fitting height at most $5d$. Therefore every finite subgroup of K/N has Fitting height at most $5d$. Therefore (by Corollary 3.10 of [4], for example) K/N has a series of length $5d$ with factors which are locally nilpotent. Therefore K/N is π-separable for all π. It follows that $O_{p,p',p}(G)$ is π-separable for all π. But, by Theorem B of [1], the Sylow p-subgroups of every subgroup of G are conjugate. Therefore, by Theorem A of [7], $G/O_{p,p',p}(G)$ is finite. Therefore G is π-separable for all π.

By Lemma 1.5, it follows that $G \in \mathfrak{U}$. Hence $G/\rho(G)$ is soluble by Theorem E of [5]. But $\rho(G)$ is soluble by Corollary 2.3 of [1]. Hence G is soluble.

3. A Counterexample

If G is a periodic linear group then, for all but finitely many primes p, the Sylow p-subgroups of G are Abelian: see Corollary 9.4 of [13]. One might well conjecture that a similar result holds for periodic soluble $\mathfrak{M}_c$-groups because of the similarity with linear groups shown by Theorem 2.1. But we now describe an example of a periodic metabelian $\mathfrak{M}_c$-group with non-Abelian Sylow p-subgroups for all primes p. For brevity, some routine details are omitted.

For each prime p, let A_p be a cyclic group of order p. Let A be the restricted direct product of the A_p. For each prime p, let K_p be a field of characteristic p containing a primitive qth root of unity for each prime $q \neq p$, and let B_p be a p-dimensional vector space over K_p. Let B be the restricted direct sum of the additive groups B_p. For each p, A may be represented faithfully as a group of linear transformations of B_p in which A_p permutes a basis of B_p and, for $q \neq p$, A_q is represented by scalars. Hence we may form a semidirect product $G = AB$. Clearly, G is a periodic metabelian group with non-Abelian Sylow p-subgroups for all primes p. It remains to show that $G \in \mathfrak{M}_c$.

An element a of A will be called "prime" if $a \in A_p \backslash 1$ for some p, and "composite" if $a \notin A_p$ for all p. For each $a \in A$ let θ_a be the map from B to B defined by $b\theta_a = [b, a]$ for all $b \in B$. It may be verified that if $a \in A\backslash A_p$, then θ_a restricts to a nonsingular linear transformation of B_p. Hence, if a is composite, θ_a is an automorphism of B.

To obtain a contradiction, suppose that $G \notin \mathfrak{M}_c$. Then it is easy to verify that there exists an infinite sequence

$$(g_1, h_1), (g_2, h_2), (g_3, h_3),...$$

of pairs of elements of G such that, for all n, h_n centralizes $g_1, g_2,..., g_{n-1}$ but not g_n. We write $g_n = a_n b_n$ and $h_n = \alpha_n \beta_n$, where $a_n, \alpha_n \in A$ and $b_n, \beta_n \in B$.

By passing to an infinite subsequence we may assume that $a_n = 1$ for all n or a_n is prime for all n or a_n is composite for all n. If a_n is prime for all n, then by passing to an infinite subsequence we may assume that either $a_i = a_j$ for all i, j or $a_i^{-1} a_j$ is composite for all i, j ($i \neq j$), and we may pass to the sequence

$$(g_1^{-1} g_2, h_2), (g_1^{-1} g_3, h_3), (g_1^{-1} g_4, h_4),... .$$

Thus we may assume that $a_n = 1$ for all n or a_n is composite for all n. Similarly, by passing to an infinite subsequence or the sequence

$$(g_1, h_1^{-1} h_2), (g_2, h_2^{-1} h_3), (g_3, h_3^{-1} h_4),...,$$

we may assume that $\alpha_n = 1$ for all n or α_n is composite for all n. Thus there are the following four cases to consider.

(i) $a_n = \alpha_n = 1$ for all n.

(ii) a_n is composite and $\alpha_n = 1$ for all n.

(iii) $a_n = 1$ and α_n is composite for all n.

(iv) a_n and α_n are composite for all n.

Clearly (i) contradicts $[g_1, h_1] \neq 1$. In case (ii) we have

$$1 = [h_2, g_1] = [\beta_2, a_1] = \beta_2 \theta_{a_1}.$$

Thus $\beta_2 = 1$, since a_1 is composite, contradicting $[g_2, h_2] \neq 1$. Case (iii) is similar. It remains to consider case (iv). Suppose $1 \leqslant i < j$. Then from $[a_i b_i, \alpha_j \beta_j] = 1$ we deduce $[b_i, \alpha_j] = [\beta_j, a_i]$, that is, $b_i \theta_{\alpha_j} = \beta_j \theta_{a_i}$. Since a_i and α_j are composite and θ_{a_i} and θ_{α_j} commute we may write $b_i \theta_{a_i}^{-1} = \beta_j \theta_{\alpha_j}^{-1}$. Thus

$$\beta_2 \theta_{\alpha_2}^{-1} = b_1 \theta_{a_1}^{-1} = \beta_3 \theta_{\alpha_3}^{-1} = b_2 \theta_{a_2}^{-1},$$

giving $b_2 \theta_{\alpha_2} = \beta_2 \theta_{a_2}$ and contradicting $[g_2, h_2] \neq 1$.

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